

# INTEGRAL VAN VLECK'S AND KANNAPPAN'S FUNCTIONAL EQUATIONS ON SEMIGROUPS

ELQORACHI ELHOUCIEN

ABSTRACT. In this paper we study the solutions of the integral Van Vleck's functional equation for the sine

$$\int_S f(x\tau(y)t)d\mu(t) - \int_S f(xyt)d\mu(t) = 2f(x)f(y), \quad x, y \in S$$

and the integral Kannappan's functional equation

$$\int_S f(xyt)d\mu(t) + \int_S f(x\tau(y)t)d\mu(t) = 2f(x)f(y), \quad x, y \in S,$$

where  $S$  is a semigroup,  $\tau$  is an involution of  $S$  and  $\mu$  is a measure that is linear combinations of point measures  $(\delta_{z_i})_{i \in I}$ , such that for all  $i \in I$ ,  $z_i$  is contained in the center of  $S$ .

We express the solutions of the first equation by means of multiplicative functions on  $S$ , and we prove that the solutions of the second equation are closely related to the solutions of the classic d'Alembert's functional equation with involution.

## 1. INTRODUCTION

Throughout this paper  $S$  denotes a semigroup, and  $\tau : S \rightarrow S$  is an involution of  $S$ . That is  $\tau(xy) = \tau(y)\tau(x)$  and  $\tau(\tau(x)) = x$  for all  $x, y \in S$ . If  $\mu$  denotes a discrete complex measure, we say that  $\mu$  is  $\tau$ -invariant and we write  $\mu = \tau(\mu)$  if  $\int_S f(\tau(t))d\mu(t) = \int_S f(t)d\mu(t)$  for all complex-valued continuous and bounded function  $f$  on a topological semigroup  $S$ . A function  $\chi : S \rightarrow \mathbb{C}$  is a multiplicative function if  $\chi(xy) = \chi(x)\chi(y)$  for all  $x, y \in S$ .

In 2003, Elqorachi and Akkouchi [4] introduced and studied the bounded and continuous solutions  $f \neq 0$  of the following generalized d'Alembert's functional equation

$$(1.1) \quad \int_G f(xty)d\mu(t) + \int_G f(xt\tau(y))d\mu(t) = 2f(x)f(y), \quad x, y \in G$$

on a topological group  $G$ . They proved that under the conditions that  $\mu = \tau(\mu)$  and  $f$  satisfies the Kannappan's condition

$\int_G \int_G f(xtysz)d\mu(t)d\mu(s) = \int_G \int_G f(ytxsz)d\mu(t)d\mu(s)$ ,  $x, y, z \in G$ , there exists a generalized  $\mu$ -spherical function  $\psi : G \rightarrow \mathbb{C}$ :

$$(1.2) \quad \int_G \psi(xty)d\mu(t) = \psi(x)\psi(y), \quad x, y \in G$$

such that  $f(x) = \frac{\psi(x) + \psi(\tau(x))}{2}$  for all  $x \in G$ .

$\mu$ -spherical function and related topics are studied in [1, 2]

---

*Key words and phrases.* semigroup; d'Alembert's equation; Van Vleck's equation; Kannappan's equation; involution; multiplicative function; complex measure.

2010 Mathematics Subject Classification: 39B32, 39B52.

In the particular case when  $\mu = \delta_{z_0}$  is the Dirac measure, the functional equation (1.1) reduces to Kannappan's functional equation [5]

$$(1.3) \quad f(xz_0y) + f(xz_0\tau(y)) = 2f(x)f(y), \quad x, y \in S.$$

Kannappan proved that any solution  $f: \mathbb{R} \rightarrow \mathbb{C}$  of (1.3) with  $\tau(y) = -y$  for all  $y \in \mathbb{R}$  is periodic, if  $z_0 \neq 0$ . Furthermore, the periodic solutions has the form  $f(x) = g(x - z_0)$  where  $g$  is a periodic solution of d'Alembert functional equation

$$(1.4) \quad g(x + y) + g(x - y) = 2g(x)g(y), \quad x, y \in \mathbb{R}.$$

Perkins and Sahoo [6] studied the following version of Kannappan's functional equation

$$(1.5) \quad f(xy z_0) + f(xy^{-1} z_0) = 2f(x)f(y), \quad x, y \in S$$

on groups. They found the form of any abelian solution  $f$  of (1.5).

Recently, Stetkær [8] took  $z_0$  in the center and expressed the complex-valued solutions of Kannappan's functional equation (1.3) on semigroups in terms of solutions of d'Alembert's functional equation

$$(1.6) \quad g(xy) + g(x\tau(y)) = 2g(x)g(y), \quad x, y \in S,$$

The complex-valued solutions of (1.6) are formulated by Davison [3] on monoids that need not be commutative.

Stetkær [9, Exercise 9.18] found the complex-valued solution of Van Vleck's functional equation

$$(1.7) \quad f(xy^{-1} z_0) - f(xy z_0) = 2f(x)f(y), \quad x, y \in G,$$

when  $G$  is a not necessarily abelian group and  $z_0$  is a fixed element in the center of  $G$ . We refer also to [11] and [12].

Perkins and Sahoo [6] replaced the group inversion by an involution  $\tau: G \rightarrow G$  and they obtained the abelian, complex-valued solutions of equation

$$(1.8) \quad f(x\tau(y) z_0) - f(xy z_0) = 2f(x)f(y), \quad x, y \in G.$$

Stetkær [7] extends the results of Perkins and Sahoo [6] about equation (1.7) to the case where  $G$  is a semigroup and the solutions are not assumed to be abelian.

The main purpose of this paper is to extend Stetkær's results [7, 8] to the following generalizations of Van Vleck's functional equation for the sine

$$(1.9) \quad \int_S f(x\tau(y)t) d\mu(t) - \int_S f(xyt) d\mu(t) = 2f(x)f(y), \quad x, y \in S,$$

and of Kannappan's functional equation

$$(1.10) \quad \int_S f(xyt) d\mu(t) + \int_S f(x\tau(y)t) d\mu(t) = 2f(x)f(y), \quad x, y \in S,$$

where  $S$  is a semigroup,  $\tau$  is an involution of  $S$  and  $\mu$  is a measure that is linear combinations of point measures  $(\delta_{z_i})_{i \in I}$ , with  $z_i$  contained in the center of  $S$ , for all  $i \in I$ .

We express the solutions of (1.9) in terms of multiplicative functions on  $S$  and we prove that the solutions of (1.10) are closely related to the solutions of the classic d'Alembert's functional equation (1.6).

## 2. THE SOLUTIONS OF THE INTEGRAL VAN VLECK'S FUNCTIONAL EQUATION ON SEMIGROUPS

In this section we obtain the complex-valued solutions of the integral Van Vleck's functional equation (1.9) on semigroups. The following lemmas will be used later. They are generalizations of Stetkær's lemmas obtained in [7] for  $\mu = \delta_{z_0}$ , where  $z_0$  is a fixed element in the center of the semigroup  $S$ .

**Lemma 2.1.** *Let  $S$  be a semigroup with an involution  $\tau: S \rightarrow S$ . Let  $\mu$  be a complex measure with support contained in the center of  $S$ . Let  $f$  be a non-zero solution of equation (1.9). Then for all  $x \in S$  we have*

$$(2.1) \quad f(x) = -f(\tau(x)),$$

$$(2.2) \quad \int_S f(t) d\mu(t) \neq 0,$$

$$(2.3) \quad \int_S \int_S f(ts) d\mu(t) d\mu(s) = \int_S \int_S f(\tau(t)s) d\mu(t) d\mu(s) = 0,$$

$$(2.4) \quad \int_S \int_S f(x\tau(t)s) d\mu(t) d\mu(s) = f(x) \int_S f(t) d\mu(t),$$

$$(2.5) \quad \int_S \int_S f(xts) d\mu(t) d\mu(s) = -f(x) \int_S f(t) d\mu(t),$$

$$(2.6) \quad \int_S f(\tau(x)t) d\mu(t) = \int_S f(xt) d\mu(t).$$

*Proof.* Replacing  $y$  by  $\tau(y)$  in the functional equation (1.9) and using  $\tau(\tau(y)) = y$  we get

$$\begin{aligned} & \int_S f(xyt) d\mu(t) - \int_S f(x\tau(y)t) d\mu(t) = 2f(x)f(\tau(y)) \\ & = -\left[ \int_S f(x\tau(y)t) d\mu(t) - \int_S f(xyt) d\mu(t) \right] = -2f(x)f(y), \end{aligned}$$

which implies the formula (2.1).

By replacing  $x$  by  $\tau(s)$  in (1.9) and using (2.1) we have

$$(2.7) \quad \int_S f(\tau(s)\tau(y)t) d\mu(t) - \int_S f(\tau(s)yt) d\mu(t) = -2f(s)f(y)$$

for all  $x, s \in S$ . By integrating the two members of equation (2.7) with respect to  $s$  we obtain

$$\begin{aligned} (2.8) \quad & \int_S \int_S f(\tau(s)\tau(y)t) d\mu(s) d\mu(t) - \int_S \int_S f(\tau(s)yt) d\mu(s) d\mu(t) \\ & = -2f(y) \int_S f(s) d\mu(s). \end{aligned}$$

By using (2.1) we find

$$\int_S \int_S f(\tau(s)\tau(y)t) d\mu(s) d\mu(t) = - \int_S \int_S f(\tau(t)ys) d\mu(s) d\mu(t)$$

$$= - \int_S \int_S f(\tau(t)sy) d\mu(s) d\mu(t)$$

so, we obtain

$$-2 \int_S \int_S f(\tau(t)sy) d\mu(t) d\mu(s) = -2f(x) \int_S f(s) d\mu(s),$$

which proves (2.4).

Setting  $y = s$  in (1.9) and integrating the result obtained with respect to  $s$  we get by using (2.4) that

$$\begin{aligned} \int_S \int_S f(x\tau(s)t) d\mu(s) d\mu(t) - \int_S \int_S f(xst) d\mu(s) d\mu(t) &= 2f(x) \int_S f(s) d\mu(s) \\ &= f(x) \int_S f(s) d\mu(s) - \int_S \int_S f(xst) d\mu(s) d\mu(t). \end{aligned}$$

So, we deduce formula (2.5).

By replacing  $x$  by  $xs$  in the functional equation (1.9) and integrating the result obtained with respect to  $s$  we get by using (2.5) and the support of  $\mu$  contained in the center of  $S$  that

$$\begin{aligned} \int_S \int_S f(xs\tau(y)t) d\mu(s) d\mu(t) - \int_S \int_S f(xsy t) d\mu(s) d\mu(t) \\ = 2f(y) \int_S f(xs) d\mu(s) \\ = -f(x\tau(y)) \int_S f(s) d\mu(s) + f(xy) \int_S f(s) d\mu(s). \end{aligned}$$

If  $\int_S f(s) d\mu(s) = 0$ , then  $f(y) \int_S f(xs) d\mu(s) = 0$  for all  $x, y \in S$ . Since  $f \neq 0$  then  $\int_S f(xs) d\mu(s) = 0$  for all  $x \in S$ , so we have

$$\int_S f(x\tau(y)t) d\mu(t) - \int_S f(xyt) d\mu(t) = 0 = 2f(x)f(y)$$

for all  $x, y \in S$  from which we deduce that  $f(x) = 0$  for all  $x \in S$ . This contradicts the assumption that  $f \neq 0$  and it follows that  $\int_S f(s) d\mu(s) \neq 0$ , so, we have (2.2). From (2.5) and (2.1), we have

$$\begin{aligned} \int_S \int_S \int_S \int_S f(\tau(st)s't') d\mu(t) d\mu(s) d\mu(t') d\mu(s') \\ = - \int_S \int_S f(\tau(st)) d\mu(s) d\mu(t) \int_S f(s) d\mu(s) = \int_S \int_S f(st) d\mu(s) d\mu(t) \int_S f(s) d\mu(s). \end{aligned}$$

By setting  $x = \tau(t)s'$ ,  $y = s$  in (1.9) and integrating the result obtained with respect to  $t$  and  $s'$  we get by a computation that

$$\begin{aligned} \int_S \int_S \int_S \int_S f(\tau(t)s'\tau(s)t') d\mu(s) d\mu(t) d\mu(s') d\mu(t') \\ - \int_S \int_S \int_S \int_S f(\tau(t)s'st') d\mu(s) d\mu(t) d\mu(s') d\mu(t') \\ = 2 \int_S \int_S f(\tau(t)s') d\mu(t) d\mu(s') \int_S f(s) d\mu(s), \end{aligned}$$

which can be written as follows

$$\int_S \int_S f(st) d\mu(s) d\mu(t) \int_S f(s) d\mu(s)$$

$$\begin{aligned}
& + \int_S \int_S f(\tau(t)s) d\mu(s) d\mu(t) \int_S f(s) d\mu(s) \\
& = 2 \int_S \int_S f(\tau(t)s') d\mu(t) d\mu(s') \int_S f(s) d\mu(s).
\end{aligned}$$

This implies that

$$\int_S \int_S f(st) d\mu(s) d\mu(t) = \int_S \int_S f(\tau(t)s) d\mu(s) d\mu(t).$$

On the other hand, since  $f(\sigma(x)) = -f(x)$  for all  $x \in S$ , we get

$$\int_S \int_S f(\tau(t)s) d\mu(t) d\mu(s) = - \int_S \int_S f(\tau(t)s) d\mu(t) d\mu(s),$$

and then we obtain

$$\int_S \int_S f(\tau(t)s) d\mu(t) d\mu(s) = 0 = \int_S \int_S f(ts) d\mu(t) d\mu(s),$$

which proves (2.3).

By replacing  $x$  by  $st$  in (1.9) and integrating the result obtained with respect to  $s$  and  $t$  we get

$$\begin{aligned}
& \int_S \int_S \int_S f(st\tau(y)t') d\mu(s) d\mu(t) d\mu(t') \\
& - \int_S \int_S \int_S f(styt') d\mu(s) d\mu(t) d\mu(t') \\
& = 2f(y) \int_S \int_S f(st) d\mu(t) d\mu(s) = 0.
\end{aligned}$$

From (2.5) we have

$$\begin{aligned}
& \int_S \int_S \int_S f(st\tau(y)t') d\mu(s) d\mu(t) d\mu(t') \\
& = \int_S \left[ \int_S \int_S f(\tau(y)t'ts) d\mu(s) d\mu(t) \right] d\mu(t') \\
& = - \int_S f(s) d\mu(s) \int_S \int_S f(\tau(y)t') d\mu(t')
\end{aligned}$$

and

$$\begin{aligned}
& \int_S \int_S \int_S f(styt') d\mu(s) d\mu(t) d\mu(t') d\mu(s) \\
& = \int_S \left[ \int_S \int_S f(yt'st) d\mu(s) d\mu(t) \right] d\mu(t') \\
& = - \int_S f(s) d\mu(s) \int_S f(yt') d\mu(t').
\end{aligned}$$

Since  $\int_S f(s) d\mu(s) \neq 0$  we deduce that  $\int_S f(yt') d\mu(t') = \int_S f(\tau(y)t') d\mu(t')$  for all  $y \in S$ . This completes the proof.  $\square$

**Lemma 2.2.** *Let  $f$  be a non-zero solution of equation (1.9). Then (1) the function defined by*

$$g(x) := \frac{\int_S f(xt) d\mu(t)}{\int_S f(s) d\mu(s)} \text{ for } x \in S$$

is a non-zero abelian solution of d'Alembert's functional equation (1.6).

(2)

$$\int_S \int_S g(ts) d\mu(t) d\mu(s) \neq 0; \quad \int_S g(s) d\mu(s) = 0$$

(3) The function  $g$  from (1) has the form  $g = \frac{\chi + \chi \circ \tau}{2}$ , where  $\chi : S \rightarrow \mathbb{C}$ ,  $\chi \neq 0$ , is a multiplicative function on  $S$ .

*Proof.* (1). From (2.4), (2.5) and the definition of  $g$  we get by a computation that

$$\begin{aligned} & \left( \int_S f(s) d\mu(s) \right)^2 [g(xy) + g(x\tau(y))] = \\ & \int_S f(s) d\mu(s) \int_S f(xyt) d\mu(t) + \int_S f(s) d\mu(s) \int_S f(x\tau(y)t) d\mu(t) \\ & = - \int_S \int_S \int_S f(xytss') d\mu(t) d\mu(s) d\mu(s') \\ & \quad + \int_S \int_S \int_S f(x\tau(y)t\tau(s)s') d\mu(t) d\mu(s) d\mu(s') \\ & = \int_S \int_S \int_S f(xs'\tau(y)s)t d\mu(t) d\mu(s) d\mu(s') - \int_S \int_S \int_S f(xs'yst) d\mu(t) d\mu(s) d\mu(s') \\ & = 2 \int_S f(xs') d\mu(s') \int_S f(ys) d\mu(s) \end{aligned}$$

which implies the desired result.

(2). From (2.5) and the definition of  $g$  we get

$$\begin{aligned} \int_S \int_S g(ts) d\mu(t) d\mu(s) &= \frac{\int_S \int_S \int_S f(s'ts) d\mu(t) d\mu(s) d\mu(s')}{\int_S f(s) d\mu(s)} \\ &= \frac{- \int_S f(s') d\mu(s') \int_S f(s) d\mu(s)}{\int_S f(s) d\mu(s)} = - \int_S f(s) d\mu(s) \neq 0. \end{aligned}$$

From (2.3) and the definition of  $g$  we get

$$\int_S g(s) d\mu(s) = \frac{\int_S f(st) d\mu(s) d\mu(t)}{\int_S f(s) d\mu(s)} = \frac{0}{\int_S f(s) d\mu(s)} = 0$$

Furthermore,  $\int_S \int_S g(st) d\mu(t) d\mu(s) \neq 0$ , so  $g \neq 0$ .

As  $g$  is a solution of equation (1.6) then by [9, Proposition 9.17(c)]  $g$  is a solution of pre-d'Alembert function. Now, according to [9, Proposition 8.14(a)] we discuss two cases:

**Case 1.** If there is a  $t$  in the center of  $S$  such that  $g(t)^2 \neq d(t)$ , then  $g$  is abelian.

**Case 2** If for all  $t$  in the center of  $S$  satisfies  $g(t)^2 = d(t)$ , then we get  $g(xt) = g(x)g(t)$  for all  $x \in S$  and  $t$  in the center of  $S$ . By integrating the expression with respect to  $t$  we get  $\int_S g(xt) d\mu(t) = g(x) \int_S g(t) d\mu(t) = 0$  for all  $x \in S$  and then  $\int_S \int_S g(st) d\mu(t) d\mu(s) = 0$ . This contradicts the first assertion of Lemma 2.2 (2). Finally, we conclude that  $g$  is abelian and for the rest of the proof we use [9, Theorem 9.12].  $\square$

The main content of this section is the following theorem.

**Theorem 2.3.** *The non-zero solutions  $f : S \longrightarrow \mathbb{C}$  of the functional equation (1.9) are the functions of the form*

$$(2.9) \quad f = \left[ \frac{\chi - \chi \circ \tau}{2} \right] \int_S \chi(\tau(t)) d\mu(t),$$

where  $\chi : S \longrightarrow \mathbb{C}$  is a multiplicative function such that  $\int_S \chi(t) d\mu(t) \neq 0$  and  $\int_S \chi(\tau(t)) d\mu(t) = -\int_S \chi(t) d\mu(t)$ .

If  $S$  is a topological semigroup and that  $\tau : S \longrightarrow S$ , is continuous, then the non-zero solution  $f$  of equation (1.9) is continuous, if and only if  $\chi$  is continuous.

*Proof.* Let  $f : S \longrightarrow \mathbb{C}$ ,  $f \neq 0$ , be a solution of equation (1.9). Then by replacing  $y$  by  $s$  and integrating the result obtained with respect to  $s$  we get

$$\begin{aligned} f(x) &= \frac{\int_S \int_S f(x\tau(s)t) d\mu(s) d\mu(t) - \int_S \int_S f(xst) d\mu(s) d\mu(t)}{2 \int_S f(s) d\mu(s)} \\ &= \frac{\int_S g(x\tau(s)) d\mu(s) - \int_S g(xs) d\mu(s)}{2} \end{aligned}$$

for all  $x \in S$  and where  $g$  is the function given by Lemma 2.2(1). According to  $g = \frac{\chi + \chi \circ \tau}{2}$  we get the following formula

$$(2.10) \quad f = \left[ \frac{\int_S \chi(s) d\mu(s) - \int_S \chi(\tau(s)) d\mu(s)}{2} \right] \left[ \frac{\chi \circ \tau - \chi}{2} \right].$$

In view of Lemma 2.1 we have  $\int_S f(\tau(x)t) d\mu(t) = \int_S f(xt) d\mu(t)$  for all  $x \in S$ . Substituting (2.10) into (2.6) we find after simple computations that

$$\left[ \int_S \chi(\tau(s)) d\mu(s) + \int_S \chi(s) d\mu(s) \right] [\chi - \chi \circ \tau] = 0.$$

The rest of the proof is similar to Stetkær's proof [7].  $\square$

**Corollary 2.4.** [7] *Let  $S$  be a semigroup with an involution  $\tau : S \longrightarrow S$ . If  $\mu = \delta_{z_0}$ , where  $z_0$  is a fixed element in the center of  $S$ . The non-zero solutions  $f : S \longrightarrow \mathbb{C}$  of the functional equation (1.8) are the functions of the form*

$$(2.11) \quad f = \chi(\tau(z_0)) \left[ \frac{\chi - \chi \circ \tau}{2} \right],$$

where  $\chi : S \longrightarrow \mathbb{C}$  is a multiplicative function such that  $\chi(z_0) \neq 0$  and  $\chi(\tau(z_0)) = -\chi(z_0)$ .

### 3. INTEGRAL KANNAPPAN'S FUNCTIONAL EQUATION ON SEMIGROUPS

In this section we study the complex-valued solutions of the functional equation (1.10). The support of the discrete complex measure  $\mu$  is assumed to be contained in the center of the semigroup  $S$ .

The following useful lemma will be used later. It's a natural generalization of Lemma 1 and Lemma 2 obtained by Stetkær [8] for  $\mu = \delta_{z_0}$ .

**Lemma 3.1.** (1) *If  $f : S \longrightarrow \mathbb{C}$  is a solution of (1.10), then for all  $x \in S$  we have*

$$(3.1) \quad f(x) = f(\tau(x)),$$

$$(3.2) \quad \int_S f(t) d\mu(t) \neq 0 \iff f \neq 0.$$

$$(3.3) \quad \int_S \int_S f(x\sigma(t)s) d\mu(t) d\mu(s) = f(x) \int_S f(t) d\mu(t),$$

$$(3.4) \quad \int_S \int_S f(xts) d\mu(t) d\mu(s) = f(x) \int_S f(t) d\mu(t),$$

(2) If  $g: S \rightarrow \mathbb{C}$  is a solution of d'Alembert's functional equation (1.6), then

(i)  $g(x) = g(\tau(x))$  for all  $x \in S$ .

(ii) The following properties are equivalent

$$(3.5) \quad \int_S g(xt) d\mu(t) = \int_S g(x\tau(t)) d\mu(t) \text{ for all } x \in S$$

$$(3.6) \quad \int_S g(xt) d\mu(t) = g(x) \int_S g(t) d\mu(t) \text{ for all } x \in S$$

$$(3.7) \quad \int_S \int_S g(ts) d\mu(t) d\mu(s) = \left( \int_S g(t) d\mu(t) \right)^2$$

*Proof.* (1). The formula (3.1) is proved like the corresponding statement in Lemma 2.1.

By putting  $x = \tau(s)$  in (1.10) and integrating the result obtained with respect to  $s$  to get

$$\begin{aligned} & \int_S \int_S f(\tau(s)yt) d\mu(t) d\mu(s) \\ & + \int_S \int_S f(\tau(s)\tau(y)t) d\mu(t) d\mu(s) = 2f(y) \int_S f(\tau(t)) d\mu(t) = 2f(y) \int_S f(t) d\mu(t), \end{aligned}$$

where the last equality holds, because  $f$  satisfies (3.1).

In view of (3.1), we have

$$\int_S \int_S f(\tau(s)\tau(y)t) d\mu(t) = \int_S \int_S f(\tau(t)ys) d\mu(t) d\mu(s).$$

So, we obtain  $2 \int_S \int_S f(\tau(t)ys) d\mu(t) d\mu(s) = 2f(y) \int_S f(t) d\mu(t)$ , which proves (3.3).

By setting  $y = s$  in (1.10) and integrating the result obtained with respect to  $s$  we get

$$\begin{aligned} & \int_S \int_S f(xst) d\mu(t) d\mu(s) \\ & + \int_S \int_S f(x\tau(s)t) d\mu(t) d\mu(s) = 2f(x) \int_S f(s) d\mu(s) \\ & = \int_S \int_S f(xst) d\mu(t) d\mu(s) \\ & + f(x) \int_S f(s) d\mu(s), \end{aligned}$$

which implies the formula (3.4).

Assume that  $f$  is a solution of equation (1.10) and that  $\int_S f(t) d\mu(t) = 0$ . Replacing  $x$  by  $xs$ ,  $y$  by  $yt$  in (1.10) and integrating the result obtained with respect to  $s$  and  $t$  we find

$$\begin{aligned} & \int_S \int_S \int_S f(xsytk) d\mu(t) d\mu(s) d\mu(k) + \int_S \int_S \int_S f(xs\tau(t)\tau(y)k) d\mu(t) d\mu(s) d\mu(k) \\ & = 2 \int_S f(xs) d\mu(s) \int_S f(yt) d\mu(t). \end{aligned}$$



Since, from (3.3) we have

$$\begin{aligned} \int_S \int_S \int_S f(xs\tau(t)\tau(y)k)d\mu(t)d\mu(s)d\mu(k) &= \int_S [\int_S \int_S f(xs\tau(y)\tau(t)k)d\mu(t)d\mu(k)]d\mu(s) \\ &= \int_S [\int_S f(t)d\mu(t)f(xs\tau(y))]d\mu(s) = \int_S 0d\mu(s) = 0. \end{aligned}$$

In view of (3.4) we have

$$\begin{aligned} \int_S \int_S \int_S f(xsytk)d\mu(t)d\mu(s)d\mu(k) &= \int_S [\int_S \int_S f(xystk)d\mu(t)d\mu(k)]d\mu(s) \\ &= \int_S [\int_S f(t)d\mu(t)f(xys)]d\mu(s) = \int_S 0d\mu(s) = 0, \end{aligned}$$

and it follows that  $\int_S f(xs)d\mu(s) \int_S f(yt)d\mu(s) = 0$  for all  $x, y \in S$ . So, we obtain

$$\int_S f(xyt)d\mu(t) + \int_S f(x\tau(y)t)d\mu(t) = 2f(x)f(y) = 0$$

for all  $x, y \in S$ . Consequently,  $f(x) = 0$  for all  $x \in S$  and this proves (3.2).

(2) Let  $g$  be a solution of (1.6). Assume that  $\int_S g(xt)d\mu(t) = \int_S g(x\tau(t))d\mu(t)$  holds for all  $x \in S$ . Since  $g(xt) + g(x\tau(t)) = 2g(x)g(t)$  for all  $x, t \in S$ , then by integrating the statement with respect to  $t$ , we get

$$\int_S g(xt)d\mu(t) + \int_S g(x\tau(t))d\mu(t) = 2g(x) \int_S g(t)d\mu(t) = 2 \int_S g(xt)d\mu(t).$$

Conversely,

$$\begin{aligned} 2 \int_S g(xt)d\mu(t) &= 2g(x) \int_S g(t)d\mu(t) = \int_S [g(xt) + g(x\tau(t))]d\mu(t) \\ &= \int_S g(xt)d\mu(t) + \int_S g(x\tau(t))d\mu(t), \end{aligned}$$

which implies that  $\int_S g(xt)d\mu(t) = \int_S g(x\tau(t))d\mu(t)$  for all  $x \in S$  and that (3.5) and (3.6) are equivalent.

Now, we will show that (3.7) and (3.6) are equivalent. If  $\int_S g(st)d\mu(t) = g(s) \int_S g(t)d\mu(t)$  for all  $s \in S$ , then by integration this expression with respect to  $s$ , we get  $\int_S \int_S g(st)d\mu(s)d\mu(t) = (\int_S g(t)d\mu(t))^2$ . Conversely, suppose that  $\int_S \int_S g(st)d\mu(s)d\mu(t) = (\int_S g(t)d\mu(t))^2$ . Since  $g$  is a solution of d'Alembert's functional equation (1.6), then  $g$  is a solution of the pre-d'Alembert functional equation [9, Proposition 9.17]. So, from [9, Proposition 8.14(a)] we will discuss the following two cases.

**Case 1.** If for all  $s$  in the center of  $S$  satisfies  $g(s)^2 = d(s)$ , then  $g(xs) = g(x)g(s)$  for all  $x \in S$ . So, by integrating this expression with respect to  $s$  we get  $\int_S g(xs)d\mu(s) = g(x) \int_S g(s)d\mu(s)$  for all  $x \in S$ .

**Case 2.** If there is  $s$  in the center of  $S$  such that  $g(s)^2 \neq d(s)$ , then  $g$  is abelian and there exists a multiplicative function  $\chi: S \rightarrow \mathbb{C}$  such that  $g = \frac{\chi + \chi \circ \tau}{2}$ . Substituting this into  $\int_S \int_S g(st)d\mu(s)d\mu(t) = (\int_S g(t)d\mu(t))^2$ , gives after an elementary computations that

$$\int_S \chi(t)d\mu(t) - \int_S \chi(\tau(t))d\mu(t) = 0.$$

Thus, we get

$$\int_S g(xt)d\mu(t) = \int_S \frac{\chi + \chi \circ \tau}{2}(xt)d\mu(t)$$

$$\begin{aligned}
&= \frac{1}{2}(\chi(x) \int_S \chi(t) d\mu(t) + \chi(\tau(x)) \int_S \chi(\tau(t)) d\mu(t)) = \int_S \chi(t) d\mu(t) \frac{\chi(x) + \chi \circ \tau(x)}{2} \\
&= g(x) \int_S g(t) d\mu(t).
\end{aligned}$$

This completes the proof.  $\square$

Now, we are ready to prove the second main result of this paper. We use the following notations [8] :

- $\mathcal{A}$  consists of the solution of  $g : S \rightarrow \mathbb{C}$  of d'Alembert's functional equation (1.6) with  $\int_S g(t) d\mu(t) \neq 0$  and satisfying the conditions of Lemma 3.1(2)(ii).
- To any  $g \in \mathcal{A}$  we associate the function  $Tg = \int_S g(t) d\mu(t) g : S \rightarrow \mathbb{C}$ .
- $\mathcal{K}$  consists of the non-zero solutions  $f : S \rightarrow \mathbb{C}$  of integral Kannappan's functional equation (1.10).

**Theorem 3.2.** (1)  $T$  is a bijection of  $\mathcal{A}$  onto  $\mathcal{K}$ . The inverse  $T^{-1} : \mathcal{K} \rightarrow \mathcal{A}$  is defined by

$$(T^{-1}f)(x) = \frac{\int_S f(xt) d\mu(t)}{\int_S f(t) d\mu(t)}$$

for all  $f \in \mathcal{K}$  and  $x \in S$ .

(2) Any non-zero solution  $f : S \rightarrow \mathbb{C}$  of integral Kannappan's functional equation (1.10) is of the form  $f = \int_S g(t) d\mu(t) g$ , where  $g \in \mathcal{A}$ . Furthermore,  $f(x) = \int_S g(xt) d\mu(t) = \int_S g(x\tau(t)) d\mu(t) = \int_S g(t) d\mu(t) g(x)$  for all  $x \in S$ .

(3)  $f$  is abelian [9] if and only if  $g$  is abelian.

(4) If  $S$  is equipped with a topology then  $f$  is continuous if and only if  $g$  is continuous.

*Proof.* If  $g \in \mathcal{A}$ , then

$$\begin{aligned}
\int_S Tg(xyt) d\mu(t) + \int_S Tg(x\tau(y)t) d\mu(t) &= \int_S g(s) d\mu(s) \left[ \int_S g(xyt) d\mu(t) + \int_S g(x\tau(y)t) d\mu(t) \right] \\
&= \int_S g(s) d\mu(s) \left[ g(xy) \int_S g(t) d\mu(t) + g(x\tau(y)) \int_S g(t) d\mu(t) \right] \\
&= \left( \int_S g(s) d\mu(s) \right)^2 [2g(xy)g(y)] = 2Tg(x)Tg(y).
\end{aligned}$$

Furthermore,  $\int_S Tg(s) d\mu(s) = \left( \int_S g(s) d\mu(s) \right)^2 \neq 0$ . So, we get  $Tg \in \mathcal{K}$ .

From [8, Lemma 3] the map  $T$  is injective. Now, we will show that  $T$  is surjective.

Let  $f \in \mathcal{K}$  and define the function

$$g(x) = \frac{\int_S f(xt) d\mu(t)}{\int_S f(t) d\mu(t)}.$$

In view of (3.3) and (3.4) we have

$$\begin{aligned}
&\left( \int_S f(s) d\mu(s) \right)^2 [g(xy) + g(x\tau(y))] \\
&= \int_S f(s) d\mu(s) \int_S f(xyt) d\mu(t) + \int_S f(s) d\mu(s) \int_S f(x\tau(y)t) d\mu(t) \\
&= \int_S \int_S \int_S f(xytst) d\mu(t) d\mu(s) d\mu(k) + \int_S \int_S \int_S f(x\tau(y)t\tau(s)k) d\mu(t) d\mu(s) d\mu(k) \\
&= \int_S \int_S \left[ \int_S f(xtyst) d\mu(k) + \int_S f(xt\tau(y)s)k) d\mu(k) \right] d\mu(t) d\mu(s) = 2 \int_S f(xt) d\mu(t) \int_S f(ys) d\mu(s)
\end{aligned}$$

$$= 2\left(\int_S f(s)d\mu(s)\right)^2 g(x)g(y).$$

It follow that  $g$  is a solution of d'Alembert's functional equation (1.6).

On the other hand, In view of (3.3) and (3.4) we have

$$\begin{aligned} \left(\int_S g(s)d\mu(s)\right)^2 &= \int_S g(s)d\mu(s) \int_S g(t)d\mu(t) = \frac{1}{2} \int_S \int_S [g(st) + g(s\tau(t))]d\mu(s)d\mu(t) \\ &= \frac{1}{2} \int_S \int_S \left[ \frac{\int_S f(stk)d\mu(k)}{\int_S f(s)d\mu(s)} + \frac{\int_S f(s\tau(t)k)d\mu(k)}{\int_S f(s)d\mu(s)} \right] d\mu(s)d\mu(t) \\ &= \frac{1}{2} \left[ \frac{\int_S \int_S \int_S f(stk)d\mu(s)d\mu(t)d\mu(k)}{\int_S f(s)d\mu(s)} \right. \\ &\quad \left. + \frac{\int_S \int_S \int_S f(s\tau(t)k)d\mu(s)d\mu(t)d\mu(k)}{\int_S f(s)d\mu(s)} \right] \\ &= \frac{1}{2} \left[ \frac{\int_S f(s)d\mu(s) \int_S f(s)d\mu(s)}{\int_S f(s)d\mu(s)} \right. \\ &\quad \left. + \frac{\int_S f(s)d\mu(s) \int_S f(s)d\mu(s)}{\int_S f(s)d\mu(s)} \right] = \int_S f(s)d\mu(s), \end{aligned}$$

and

$$\begin{aligned} \int_S g(st)d\mu(s)d\mu(t) &= \frac{\int_S \int_S \int_S f(tsk)d\mu(t)d\mu(s)d\mu(k)}{\int_S f(s)d\mu(s)} \\ &= \frac{\int_S f(t)d\mu(t) \int_S f(s)d\mu(s)}{\int_S f(s)d\mu(s)} = \int_S f(s)d\mu(s), \end{aligned}$$

which proves that  $g$  satisfies the conditions of Lemma 3.1(2)(ii).

Finally,  $(\int_S g(s)d\mu(s))^2 = \int_S f(s)d\mu(s) \neq 0$ , then  $\int_S g(s)d\mu(s) \neq 0$ . This completes the proof.  $\square$

**Corollary 3.3.** [8] If  $\mu = \delta_{z_0}$ , where  $z_0$  is a fixed element in the center of a semi group  $S$ . Then, any non-zero solution  $f: S \rightarrow \mathbb{C}$  of Kannappan's functional equation (1.3) is of the form  $f = g(z_0)g$ , where  $g$  is a solution of d'Alembert's functional equation (1.6) with  $g(z_0) \neq 0$  and satisfying the conditions of Lemma 3.1(ii).

**Proposition 3.4.** The non-zero abelian solutions of integral Kannappan's functional equation (1.10) are the functions of the form

$$f(x) = \left[ \frac{\chi(x) + \chi(\tau(x))}{2} \right] \int_S \chi(t)d\mu(t), \quad x \in S,$$

where  $\chi: S \rightarrow \mathbb{C}$  is a multiplicative function such that  $\int_S \chi(t)d\mu(t) \neq 0$  and  $\int_S \chi(\tau(t))d\mu(t) = \int_S \chi(t)d\mu(t)$ .

## REFERENCES

- [1] Akkouchi, M. and Bakali, A., Une généralisation des paires de Guelfand. Boll. Un. Math. Ital. **7** (1992), 759-822.
- [2] Akkouchi, M., Bakali, A. and Khalil, I., A Class of Functional Equations on a Locally Compact Group. J. London Math. Soc. (1998), **57** (3), 694-705.
- [3] Davison, T. M. K., D'Alembert's functional equation on topological monoids. Publ. Math. debrecen **75** 1/2 (2009), 41-66.
- [4] Elqorachi, E. and Akkouchi, M., On generalized d'Alembert and Wilson functional equations. Aequationes Math. (2003), **66**(3), 241-256
- [5] Kannappan, Pl. A functional equation for the cosine. Canad. Math. Bull. **2** (1968), 495-498.
- [6] Perkins, A.M. and Sahoo, P.K., On two functional equations with involution on groups related to sine and cosine functions. Aequationes Math. (2014). doi:10.1007/ s00010-014-0309-z.
- [7] Stetkær, H., Van Vleck's functional equation for the sine. Aequationes Math. (2014). doi:10.1007/ s00010-015-0349-z.
- [8] Stetkær, H., Kannappan's functional equation on semigroups with involution. Semigroup Forum, First online: 23 September 2015.
- [9] Stetkær, H., Functional Equations on Groups. World Scientific Publishing Co, Singapore (2013).
- [10] Stetkær, H., A variant of d'Alembert's functional equation. Aequationes Math., (2014), DOI 10.1007/s00010-014-0253-y.
- [11] Vleck Van, E.B., A functional equation for the sine. Ann. Math. Second Ser. **11** (4), 161-165 (1910).
- [12] Vleck Van, E.B., A functional equation for the sine. Additional note. Ann. Math. Second Ser. **13** (1/4), 154 (1911-1912).

IBN ZOHR UNIVERSITY, FACULTY OF SCIENCES, DEPARTMENT OF MATHEMATICS, AGADIR, MOROCCO

*E-mail address:* elqorachi@hotmail.com